BASIC DEVELOPMENTS OF FRACTIONAL CALCULUS AND ITS APPLICATIONS

A. P. Bhadane
Department of Mathematics,
Smt. Puspatai Hiray Mahila Mahavidyalaya
Malegaon Camp, Malegaon–423 205, (M. S.), India.
E-mail: ashok–bhadane@yahoo.in

and

K. C. Takale
Department of Mathematics,
RNC Arts, JDB Commerce and NSC Science College,
Nashik-Road 422 101, (M. S.) India.
E-mail: kalyanraotakale@gmail.com

Abstract

The aim of this paper is to study the basic theory and applications of fractional calculus. We have obtained fractional integral and fractional derivative of some functions. The fractional integral and fractional derivative of these functions are simulated by mathematical software Mathematica. 1

1 INTRODUCTION

In the late seventeenth century, the philosopher and creator of modern calculus G. W. Leibniz made some remarks on the meaning and possibility of fractional derivative of order $\frac{1}{2}$. In the year 1695, the well known mathematician L’Hospital was asked the question as the meaning of $\frac{d^n f}{dx^n}$, what if $n = \frac{1}{2}$ ?, that is, what is a fractional derivative ?. Leibniz (1695) answered, “This is an apparent paradox from which, one day, useful consequences will be drawn”. Thereafter, L’Hospital, Euler, Lagrange, Laplace, Riemann, Liouville developed the basic concept of fractional calculus. The rigorous investigation was first carried out by

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A. P. Bhadane & K. C. Takale

Liouville in a series of papers (1832–1837), where he defined the first outcast of an operator of fractional integration. The fractional order integral and differentiation, which represent a rapidly growing field both in theory and applications to real world problems. After this Euler took the first step in the study of fractional integration [3]. In the year 1865, Liouville and Swedish mathematician Holmgren, made an important contributions to the growing field of fractional calculus. Today there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types [8], but the Riemann-Liouville operator is still the most frequently used when fractional integration is performed [2, 5]. In the year 1967, Caputo defined a useful formula to obtain fractional derivative of the function [4, 7]. It should be noted that the appearance of the idea of fractional integration and differentiation is not any acceptable geometrical and physical interpretation of these concepts for more than 300 years. A different approach to geometric interpretation of fractional integration and fractional differentiation, based on the idea of the contact of $\alpha^{th}$ order, has been suggested by F. Ben Adda. However, it is difficult to speak about an acceptable geometrical interpretation without visualization. Finally, in the year 2002 I. Podlubny [6] represent the geometric interpretation of the integral and the fractional integral. Recently, the researchers and scientists found the use of fractional calculus in science, engineering, hydrology and finance etc.

We organize this paper as follows: In section 2, we study the some definitions of fractional integral and fractional derivative and properties of Gamma function and Mittag-Liffler function. The section 3, is devoted for the fractional integral and fractional derivative of some functions and these are simulated by mathematical software Mathematica. The section 4, is devoted for the applications of fractional calculus to Abel integral and differential equation.

2 DEFINITIONS AND PROPERTIES

In this section we discuss some definitions of fractional derivatives and fractional integrals of fractional calculus.

Definition 2.1 The gamma function is defined as follows

$$\Gamma(x) = \int_0^\infty y^{x-1}e^{-y}dy, \ x > 0$$

where for convergence of the integral, $x > 0$
Definition 2.2 \((\text{Gamma function is defined by Euler Limit)}):\)

\[
\Gamma(n) = \lim_{N \to \infty} \frac{1.2.3\ldots(N-1)}{n(n+1)(n+2)\ldots(n+N-1)} N^n
\]

\[
\Gamma(n+1) = \lim_{N \to \infty} \frac{1.2.3\ldots(N-1)}{(n+1)(n+2)\ldots(n+N)} N^{n+1}
\]

\[
\Gamma(n+1) = \lim_{N \to \infty} \frac{Nn}{n+N n(n+1)(n+2)\ldots(n+N-1)} N^n
\]

\[
= n \Gamma(n)
\]

Properties of \(\Gamma\)-Function:

(i) \(\Gamma(n+1) = n \Gamma(n), n > 0\)

(ii) \(\Gamma(n+1) = n \Gamma(n) = n(n-1) \Gamma(n-1) = \ldots = n(n-1)(n-m+1) \Gamma(n-m+1)\)

(iii) \(\Gamma(n+1) = n!, n = 1, 2, 3, \ldots\)

(iv) \(\Gamma(n) = \frac{\Gamma(n+1)}{n}, \Gamma(0) = \infty, \Gamma(1-n) = \infty, n > 0\)

(v) \(\Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin \pi n}.\)

Example 2.1 (i) \(\Gamma(\frac{1}{2}) = \sqrt{\pi}, (ii) \Gamma(\frac{1}{2} + n) = \frac{(2n)! \sqrt{\pi}}{4^n n!}, (iii) \Gamma(\frac{1}{2} - n) = \frac{(-4^n)n! \sqrt{\pi}}{(2n)!},\)

(iv) \(\Gamma(\frac{-3}{2}) = \frac{4}{3} \sqrt{\pi}, (v) \Gamma(\frac{-1}{2}) = -2 \sqrt{\pi}, (vi) \Gamma(-1) = \pm \infty\)

Definition 2.3 Let \(f(t)\) be a function of a variable \(t\) such that the function \(e^{-st}f(t)\) is integrable in \([0, \infty)\) for some domain of values of \(s\). The Laplace transform of the function \(f(t)\) is the integral is defined as \(\int_0^{\infty} e^{-st}f(t)dt\); for above domain values of \(s\). It is denoted by \(L\{f(t)\} = \int_0^{\infty} e^{-st}f(t)dt.\)

If \(L\{f(t)\} = \phi(s)\) then the inverse laplace transform of \(\phi(s)\) is defined as

\[L^{-1}\{\phi(s)\} = f(t).\]

Example 2.2 Consider \(f(t) = t^n, n\) is a positive integer. By above definition, we get

\[L\{t^n\} = \int_0^{\infty} e^{-st}t^n dt\]

\[= \left[t^n \frac{e^{-st}}{-s} - n t^{n-1} \frac{e^{-st}}{(-s)^2} + \ldots + (-1)^n n(n-1) \ldots \frac{e^{-st}}{(-s)^n} \right]_{t=0}^{t=\infty}, s > 0\]

After simplification and using Gamma function the above formula can be written as we get

\[L\{t^n\} = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}, s > 0\]

This formula gives the idea of Laplace transform of fractional calculus.
Remark 2.1 If we take $\alpha$ is non-integer order $n - 1 < \alpha \leq n$ then the Laplace transform of fractional calculus is

$$L\{t^\alpha\} = \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}}, \ n - 1 < \alpha \leq n, \ s > 0$$

Example 2.3 If $\alpha = \frac{1}{2}$, we get

$$L\left\{\frac{t^{\frac{1}{2}}}{s^2}\right\} = \frac{\sqrt{\pi}}{2s\sqrt{s}}, \ s > 0$$

Similarly, by the definition 2.3, we have obtained the Laplace transform of some functions which are given in the following table:

<table>
<thead>
<tr>
<th>Function</th>
<th>Laplace Transform</th>
<th>Function</th>
<th>Inverse Laplace transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{s^\frac{1}{2}}, \ s &gt; 0$</td>
<td>$\frac{1}{s^\frac{1}{2}}, \ s &gt; 0$</td>
<td>1</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}, \ s &gt; 0$</td>
<td>$\frac{n!}{s^{n+1}}, \ s &gt; 0$</td>
<td>$t^n$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}, \ s &gt; a$</td>
<td>$\frac{1}{s-a}, \ s &gt; a$</td>
<td>$e^{at}$</td>
</tr>
<tr>
<td>$\sin at$</td>
<td>$\frac{a}{s^2+a^2}, \ s &gt; 0$</td>
<td>$\frac{a}{s^2+a^2}, \ s &gt; 0$</td>
<td>$\sin at$</td>
</tr>
<tr>
<td>$\cos at$</td>
<td>$\frac{s}{s^2+a^2}, \ s &gt; 0$</td>
<td>$\frac{s}{s^2+a^2}, \ s &gt; 0$</td>
<td>$\cos at$</td>
</tr>
</tbody>
</table>

Definition 2.4 The Laplace transform of the differintegrable function $f(t)$ is defined for $(n-1) < \alpha \leq n$ as

$$L\left\{D^\alpha f(t)\right\} = \int_0^\infty e^{-st}0D^\alpha_t f(t)dt = s'^\alpha F(s) - \sum_{k=0}^{n-1} s'\left[0D^\alpha_{t}^{-k-1} f(t)\right]_{t=0}$$

The Laplace transform and inverse Laplace transform of some functions of fractional calculus are given in the following table:

<table>
<thead>
<tr>
<th>Function</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$t^{\frac{1}{2}}$</td>
<td>$\frac{\sqrt{\pi}}{\sqrt{s}}, s &gt; 0$</td>
<td>$\frac{1}{\sqrt{s}}$</td>
<td>$\frac{1}{\sqrt{s^2}}$</td>
</tr>
<tr>
<td>$t^{\frac{3}{2}}$</td>
<td>$\frac{2\sqrt{\pi}}{3s^{\frac{1}{2}}}, s &gt; 0$</td>
<td>$\frac{1}{s\sqrt{s}}$</td>
<td>$\frac{2}{s^2\sqrt{s}}$</td>
</tr>
<tr>
<td>$</td>
<td>\sin kt</td>
<td>$</td>
<td>$\frac{k}{\sqrt{1+k^2}}\coth\left(\frac{\pi k}{2}\right)$</td>
</tr>
</tbody>
</table>

Definition 2.5 (i) The Mittag-Leffler (1903) function of one parameter is defined as follows

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \ (\alpha \in C, \ \Re(\alpha) > 0)$$
(ii) The Mittag-Liffler (1903) function of two parameter is defined as follows

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha \in \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0) \]

**Properties:** The following are the basic properties of Mittag-Liffler function:

(I) \( E_{\alpha,\beta}(z) = zE_{\alpha,\beta}(z) + \frac{1}{\Gamma(\beta)} \), (II) \( E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z) \)

**Definition 2.6 (Grünwald-Letnikov)** The Grünwald-Letnikov definition of fractional derivative of a function generalizes the notion of backward difference quotient of integer order. In this case \( \alpha = 1 \) if the limit exists the Grünwald-Letnikov fractional derivative is the left derivative of the function. The Grünwald-Letnikov fractional derivative of order \( \alpha \) of the function \( f(x) \) is defined as [5]

\[ aD^\alpha_x f(x) = \lim_{N \to \infty} \left\{ \frac{(\frac{x-a}{N})^{-\alpha}}{\Gamma(-\alpha)} \sum_{j=0}^{N-1} \frac{\Gamma(j - \alpha)}{\Gamma(j + 1)} f(x - j \frac{x-a}{N}) \right\} \]

where \( \alpha \in \mathbb{C} \).

If \( \alpha = -1 \), we have a Riemann sum which is the first integral.

If \( \alpha = 1 \), then we have

\[ \lim_{N \to \infty} \left\{ \frac{f(x) - f(x - \frac{x-a}{N})}{\frac{x-a}{N}} \right\} \]

which is left derivative of the function \( f \) at \( x \).

**Definition 2.7 (Riemann-Liouville):** Riemann (1953), Liouville (1832)

(a) If \( f(x) \in C[a,b] \) and \( a < x < b \) then

\[ I^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t) \frac{dt}{(x-t)^{1-\alpha}} \]

where \( \alpha \in (-\infty, \infty) \)

is called the Riemann-Liouville fractional integral of order \( \alpha \).

(b) If \( f(x) \in C[a,b] \) and \( a < x < b \) then

\[ D^\alpha_{a+} f(x) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_a^x f(t) \frac{dt}{(x-t)^\alpha} \]

where \( \alpha \in (0,1) \)

is called the Riemann-Liouville fractional derivative of order \( \alpha \) [6].
Definition 2.8 (M. Caputo (1967)): If \( f(x) \in C[a, b] \) and \( a < x < b \) then the Caputo fractional derivative of order \( \alpha \) is defined as follows [6]

\[
aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \quad \text{where } n-1 < \alpha < n
\]

In the next section we obtain the fractional integral and fractional derivative of some standard functions and these are simulated by mathematical software Mathematica.

3 FRACTIONAL INTEGRAL AND FRACTIONAL DERIVATIVE OF SOME FUNCTIONS

3.1 Fractional Integral of Some Functions:

As an application of fractional integration, we can take \( \alpha = \frac{1}{2} \) which is called the semi-integral if it is used in Riemann-Liouville fractional integral formula of order \( \alpha \). Consider the Riemann-Liouville fractional integral formula

\[
I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \text{where } \alpha \in (-\infty, \infty)
\]

using this formula we have obtained the fractional integral of some functions as follows.

(i) Identity Function Let \( f(x) = x, \alpha = \frac{1}{2} \)

\[
I_0^{\frac{1}{2}} x = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \frac{t}{(x-t)^{\frac{1}{2}}} dt
\]

By putting \( x - t = u \), we get

\[
I_0^{\frac{1}{2}} x = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \frac{(x-u)}{u^{\frac{1}{2}}} du = \frac{1}{\Gamma(\frac{1}{2})} x^{\frac{3}{2}} = \frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}
\]

Similarly, we obtain the fractional integrals of identity function for different values of \( \alpha \) as follows

\[
I_0^{\frac{1}{2}} x = \frac{2x^{\frac{3}{2}}}{\sqrt{\pi}}
\]

\[
I_0^{\frac{3}{2}} x = \frac{8x^{\frac{5}{2}}}{15\sqrt{\pi}}
\]

\[
I_0^{5} x = \frac{1}{\sqrt{\pi}x}
\]

The fractional integral of identity function for \( \alpha = \frac{1}{2} \) is represented graphically by mathematical software Mathematica as follows:
(ii) **Constant Function** \( f(x) = K, \) \( K \) is constant, \( \alpha = \frac{1}{2} \)

\[
I_0^\frac{1}{2} K = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x \frac{K}{(x-t)^{\frac{1}{2}}} dt = \frac{K}{\Gamma\left(\frac{1}{2}\right)} \left[2(x-t)^{\frac{1}{2}}\right]_0^x = \frac{K}{\sqrt{2\pi}} \sqrt{x}
\]

**Unit Function:** \( f(x) = 1 \)

\[
I_0^\frac{1}{2}(1) = 2\sqrt{\frac{x}{\pi}}, \quad I_0^{-\frac{1}{2}}(1) = \sqrt{\frac{1}{\pi x}}, \quad I_0^\frac{3}{2}(1) = \frac{4x^\frac{3}{2}}{3\sqrt{\pi}}, \quad I_0^{-\frac{3}{2}}(1) = -\frac{1}{2\sqrt{\pi}} \frac{1}{x^\frac{3}{2}}
\]

The fractional integral of unit function for \( \alpha = \frac{1}{2} \) is represented graphically by mathematical software Mathematica as follows:

**Properties:** The following are the basic two properties of Mittag-Liffler function:

(i) \( E_{\alpha,\beta}(z) = z E_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)} \)  

(ii) \( E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z) \)

**Definition 3.1** \textbf{(Gr"unwald-Letnikov)} The Gr"unwald-Letnikov definition of fractional derivative of a function generalize the notion of backward difference quotient of
integer order. In this case $\alpha = 1$ if the limit exists the Grünwald-Letnikov fractional derivative is the left derivative of the function. The Grünwald-Letnikov fractional derivative of order $\alpha$ of the function $f(x)$ is define as \[5\]

$$aD_x^\alpha f(x) = \lim_{N \to \infty} \left\{ \frac{(x-a)-\alpha}{\Gamma(-\alpha)} \sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f(x-j\frac{x-a}{N}) \right\}$$

where $\alpha \in C$.

If $\alpha = -1$, we have a Riemann sum which is the first integral.

If $\alpha = 1$, then we have

$$\lim_{N \to \infty} \left\{ \frac{f(x) - f(x - \frac{x-a}{N})}{\frac{x-a}{N}} \right\}$$

which is left derivative of the function $f$ at $x$.

**Definition 3.2 (Riemann-Liouville):** Riemann (1953), Liouville (1832)

(a) If $f(x) \in C[a, b]$ and $a < x < b$ then

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} \, dt, \quad \text{where } \alpha \in (-\infty, \infty)$$

is called the Riemann-Liouville fractional integral of order $\alpha$.

(b) If $f(x) \in C[a, b]$ and $a < x < b$ then

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} \, dt, \quad \text{where } \alpha \in (0, 1)$$

is called the Riemann-Liouville fractional derivative of order $\alpha$ \[6\].

**Definition 3.3 (M. Caputo (1967)):** If $f(x) \in C[a, b]$ and $a < x < b$ then the Caputo fractional derivative of order $\alpha$ is defined as follows \[6\]

$$aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} \, d\tau, \quad \text{where } n-1 < \alpha < n$$

4 FRACTIONAL INTEGRAL AND DERIVATIVE

In this section we obtain the fractional integral and fractional derivative of some standard functions and these are simulated by mathematical software Mathematica.
4.1 Fractional Integral of Some Functions

As an application of fractional integration, we can take $\alpha = \frac{1}{2}$ which is called the semi-integral if it is used in Riemann-Liouville fractional integral formula of order $\alpha$. Consider the Riemann-Liouville fractional integral formula

$$I_\alpha^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} \, dt, \quad \text{where} \quad \alpha \in (-\infty, \infty)$$

using this formula we have obtain the fractional integral of some functions a follows.

(i) Identity Function Let $f(x) = x, \alpha = \frac{1}{2}$

$$I_{\frac{1}{2}}^{\frac{1}{2}} x = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x \frac{t}{(x-t)^{\frac{1}{2}}} \, dt$$

By putting $x-t = u$, we get

$$I_{\frac{1}{2}}^{\frac{1}{2}} x = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x \frac{u^{\frac{1}{2}}}{u^{\frac{1}{2}}} \, du = \frac{1}{\Gamma\left(\frac{3}{2}\right)} x^{\frac{3}{2}} = \frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}$$

The fractional integral of identity function for $\alpha = \frac{1}{2}$ is represented graphically by mathematical software Mathematica as follows:

![Graph of fractional integral of identity function for $\alpha = \frac{1}{2}$](image)

Similarly, we obtain the fractional integrals of identity function for different values of $\alpha$ as follows

$$I_{\frac{1}{2}}^{\frac{1}{2}} x = \frac{2x^{\frac{3}{2}}}{\sqrt{\pi}}, \quad I_{\frac{3}{2}}^{\frac{1}{2}} x = \frac{8x^{\frac{5}{2}}}{15\sqrt{\pi}}, \quad I_{\frac{3}{2}}^{\frac{3}{2}} x = \frac{1}{\sqrt{\pi}x}$$

The fractional integrals of identity function can be obtained for $\alpha = \frac{1}{2}, \frac{3}{2}$ and $\frac{3}{2}$ respectively and can be represented graphically using Mathematica software.
(ii) **Constant Function** \( f(x) = K, K \) is constant, \( \alpha = \frac{1}{2} \)

\[
I_{\alpha}^{x} K = \frac{1}{\Gamma(\frac{1}{2})} \int_{0}^{x} \frac{K}{(x - t)^{\frac{1}{2}}} \, dt = \frac{K}{\Gamma(\frac{1}{2})} [2(x - t)^{\frac{1}{2}}]_{0}^{x} = \frac{K}{\Gamma(\frac{1}{2})} \sqrt{x}
\]

**Unit Function:** \( f(x) = 1, \quad I_{\alpha}^{\frac{1}{2}} (1) = 2\sqrt{\frac{x}{\pi}} \)

The fractional integral of unit function for \( \alpha = \frac{1}{2} \) is represented graphically by mathematical software Mathematica as follows:

\[
I_{\frac{1}{2}}^{1} (1) = \sqrt{\frac{1}{\pi x}}, \quad I_{\frac{1}{2}}^{\frac{3}{2}} (1) = \frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}, \quad I_{\frac{3}{2}}^{\frac{3}{2}} (1) = -\frac{1}{2\sqrt{\pi} x^{\frac{3}{2}}}
\]

(iii) **Exponential Function:** The general formula for fractional integral of exponential function \( f(x) = e^{bx} \) is

\[
I_{\alpha}^{x} (e^{bx}) = b^{-\alpha} e^{bx}, \quad a = -\infty, \quad \alpha > 0, \quad \text{Re}(b) > 0.
\]

The graphical representation of fractional integral of exponential function \( f(x) = e^{bx} \) for \( \alpha = \frac{3}{2} \) is as follows:
4.2 Fractional Derivative of Some Functions

(i) Identity Function: (a) Apply the Grünwald-Letnikov definition to the identity function \( f(x) = x \), we get

\[
0D_0^\alpha x = \lim_{N \to \infty} \left\{ \left( \frac{N}{x} \right)^\alpha \sum_{j=0}^{N-1} \frac{\Gamma(j - \alpha)}{\Gamma(-\alpha)\Gamma(j + 1)} \left( x - j \left( \frac{x}{N} \right) \right) \right\}
= x^{1-\alpha} \left[ \lim_{N \to \infty} \left\{ N^\alpha \sum_{j=0}^{N-1} \frac{\Gamma(j - \alpha)}{\Gamma(-\alpha)\Gamma(j + 1)} \right\} - \lim_{N \to \infty} \left\{ N^{\alpha-1} \sum_{j=0}^{N-1} j \frac{\Gamma(j - \alpha)}{\Gamma(-\alpha)\Gamma(j + 1)} \right\} \right]
= \left[ \frac{1}{\Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(2-\alpha)} \right] x^{1-\alpha}
\]

Using Gamma property (i) and after simplification, we get \(0D_0^\alpha x = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)}\). Therefore, Grünwald-Letnikov fractional derivative of \( f(x) = x \) for \( \alpha = \frac{1}{2} \) is

\[
D_0^{\frac{1}{2}} x = 2\sqrt{\frac{x}{\pi}}
\]  

(4.1)

Similarly, we can obtain the fractional derivatives for other non-integer values of \( \alpha \).

(b) As an application of fractional derivative, we use Riemann-Liouville fractional derivative formula of order \( \alpha \).

\[
D_0^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} \, dt
\]
Identity Function \( f(x) = x, \ \alpha = \frac{1}{2} \)

\[
D_{0}^{\frac{1}{2}}x = \frac{1}{\Gamma(1/2)} \frac{d}{dx} \int_{0}^{x} \frac{t}{(x-t)^{1/2}} \, dt = \frac{1}{\Gamma(1/2)} \frac{d}{dx} \int_{0}^{x} \frac{t}{(x-t)^{1/2}} \, dt
\]

\[
= \frac{d}{dx} \left[ \frac{1}{\Gamma(1/2)} \int_{0}^{x} \frac{t}{(x-t)^{1/2}} \, dt \right] = \frac{d}{dx} \left[ f_{0}^{1/2}(x) \right]
\]

\[
= \frac{d}{dx} \left[ \frac{4x^2}{3\sqrt{\pi}} \right] = 2\sqrt{\frac{x}{\pi}}
\]

(4.2)

Similarly, we obtain \( D_{0}^{-\frac{1}{2}}x = \frac{4x^\frac{3}{2}}{3\sqrt{\pi}} \)

The fractional derivative of identity function for \( \alpha = \frac{1}{2} \) is represented graphically by mathematical software Mathematica as follows:

\[
D_{0}^{\frac{3}{2}}x = \frac{1}{\sqrt{\pi x}} \quad D_{0}^{\frac{5}{2}}x = \frac{8x^\frac{5}{2}}{15\sqrt{\pi}}
\]

and can be represented graphically using Mathematica software

(c) Here we obtain the first fractional derivative of the identity function \( f(x) = x \) by Caputo fractional derivative formula for different values of \( \alpha \). Consider the Caputo fractional derivative formula

\[
aD_{a}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \frac{f^{(n)}(\tau)}{(x-\tau)^{n-\alpha+1}} \, d\tau, \quad \text{where} \ n-1 < \alpha < n
\]
Putting $\alpha = \frac{1}{2}$, we get

\[ 0D_0^{\frac{1}{2}} f(x) = \frac{1}{\Gamma(1 - \frac{1}{2})} \int_0^x \frac{1}{(x - \tau)^{\frac{1}{2}}} \, d\tau = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \frac{1}{(x - \tau)^{\frac{1}{2}}} \, d\tau = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x (x - \tau)^{-\frac{1}{2}} \, d\tau = \frac{1}{\Gamma(\frac{1}{2})} \left[-2(x - \tau)^{\frac{1}{2}}\right]_0^x = 2 \frac{\sqrt{x}}{\pi} \]

Similarly, we obtain

\[ D_0^{\frac{1}{2}} x = \frac{4x^\frac{3}{2}}{3\sqrt{\pi}}, \quad D_0^{\frac{3}{2}} x = \frac{1}{\sqrt{\pi}x}, \quad D_0^{-\frac{3}{2}} x = \frac{8x^\frac{5}{2}}{15\sqrt{\pi}} \]

The fractional derivatives of the function $f(x) = x$ obtained by Grünwald-Letnikov, Riemann-Liouville and Caputo fractional derivative formula are same follow from equations (3.1) – (3.3).

(ii) Constant Function $f(x) = C$, $C$ is constant

(a) Let $f(x) = 1$, using Grünwald-Letnikov definition we have obtain the fractional derivatives of unit function as follows:

\[ 0D_x^\alpha (1) = \lim_{N \to \infty} \left\{ \left( \frac{N}{x} \right)^\alpha \sum_{j=0}^{N-1} \frac{\Gamma(j - \alpha)}{\Gamma(-\alpha)\Gamma(j + 1)} (1) \right\} \]

Using properties (1.3.14) and (1.3.18) (page no. 20, [5]), we get

\[ 0D_x^\alpha (1) = \frac{x^{-\alpha}}{\Gamma(1 - \alpha)} \]

Putting the different values of $\alpha$, we get

\[ D_0^{\frac{1}{2}} 1 = \frac{1}{\sqrt{\pi}x}, \quad D_0^{\frac{3}{2}} 1 = 2 \frac{\sqrt{x}}{\pi}, \quad D_0^{\frac{3}{2}} 1 = \frac{-1}{2\sqrt{\pi}x^2}, \quad D_0^{-\frac{3}{2}} 1 = \frac{4x^\frac{5}{2}}{3\sqrt{\pi}} \]

(b) The fractional derivative of the constant function is obtained by Riemann-Liouville definition as follows:

\[ D_0^\alpha (x) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_0^x \frac{C}{(x - t)^\alpha} \, dt = \frac{C}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_0^x (x - t)^{-\alpha} \, dt \]

\[ = \frac{C}{\Gamma(1 - \alpha)} \frac{d}{dx} \left[ -\frac{x - t^{-\alpha + 1}}{(-\alpha + 1)} \right] = \frac{C}{\Gamma(1 - \alpha)} \left( 1 - \alpha \right) x^{-\alpha} \]

The fractional derivative of unit function for $\alpha = \frac{1}{2}$ is represented graphically by mathematical software Mathematica as follows:
In particular, let $f(x) = 1$ (Unit function), the Riemann-Liouville fractional derivatives for different values of $\alpha$ are obtained as follows:

$$D^\frac{1}{2}_0 1 = \frac{1}{\sqrt{\pi}x}, \quad D^\frac{3}{2}_0 1 = 2 \sqrt{\frac{x}{\pi}}, \quad D^\frac{3}{2}_0 1 = -\frac{1}{2\sqrt{\pi}x^\frac{3}{2}}, \quad D^\frac{3}{2}_0 1 = \frac{4x^\frac{3}{2}}{3\sqrt{\pi}}$$

(c) The fractional derivative of the constant function is obtained by Caputo definition as follows:

$$aD^\alpha_C x = \frac{1}{\Gamma(n-\alpha)} \int_a^x 0 \frac{d\tau}{(x-\tau)^{n-\alpha+1}}, \quad n-1 < \alpha < n$$

**Remark 4.1** The fractional derivative of constant by Riemann-Liouville is not constant but the Caputo fractional derivative of constant is zero.

Graphical representation of the Riemann-Liouville fractional derivatives of some other standard functions for different values of $\alpha$ such as $\alpha = \frac{1}{2}, \alpha = -\frac{1}{2}, \alpha = \frac{3}{2}$ and $\alpha = -\frac{3}{2}$ are given as follows:

(iii) **Sine Function** If $f(x) = \sin(x)$ then its fractional derivative is

$$D^\alpha_0 f(x) = \sin(x + \frac{\alpha\pi}{2})$$

and it is represented graphically for $\alpha = \frac{3}{2}$ by Mathematica as follows:
(iv) Cosine Function If \( f(x) = \cos(x) \) then the fractional derivative of cosine function is

\[
D_0^\alpha f(x) = \cos\left(x + \frac{\alpha \pi}{2}\right)
\]

and it is represented graphically for \( \alpha = \frac{3}{2} \) by Mathematica as follows:

![Graph of cosine function with \( \alpha = \frac{3}{2} \)]

5 APPLICATIONS

(i) Abel’s Integral: The Abel’s fractional order integral equation is defined as [6]

\[
\frac{1}{\Gamma(\alpha)} \int_0^t \frac{\varphi(\tau)d\tau}{(t-\tau)^{1-\alpha}} = f(t), \quad (t > 0)
\]  

where \( 0 < \alpha < 1 \). The solution is given by the well-known formula [1]

\[
\varphi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^\alpha}, \quad (t > 0)
\]

In terms of fractional order derivatives, equations (5.1) and (5.2) takes on the form

\[
oD_t^{-\alpha}\varphi(t) = f(t), \quad (t > 0)\quad oD_t^\alpha f(t) = \varphi(t).
\]

Example 5.1 Let us consider the equation

\[
\int_0^\infty \frac{\varphi(\sqrt{s^2 + y^2})ds}{\sqrt{s^2 + y^2}} = f(y) \frac{1}{2y}
\]  

After simplification, we arrive at an equation of the type (5.1), with \( \alpha = \frac{1}{2} \)

\[
\int_0^t \frac{\varphi(\tau)d\tau}{(t-\tau)^{1/2}} = f\left(\frac{1}{\sqrt{t}}\right)
\]

The solution of equation (5.5) can be found with help of formula (5.4) as follows

\[
\varphi(t) = \frac{1}{\Gamma(1/2)} oD_{1/2}^1 f\left(\frac{1}{\sqrt{t}}\right)
\]
and performing backward substitution we obtain the solution of equation (5.5) in terms of fractional derivatives
\[
\varphi(\frac{1}{\sqrt{t}}) = \frac{t}{\sqrt{\pi}} \, _0D_t^{1/2}f(\frac{1}{\sqrt{t}})
\]  
(5.7)

(ii) Differential Equation

**Example 5.2** Consider the ordinary differential equation
\[
_0D_t^\alpha f(t) + af(t) = 0, (t > 0), \left[ _0D_t^{\alpha-1}f(t) \right]_{t=0} = C, (n-1 < \alpha \leq n).
\]
Applying the Laplace transform on both sides, we get
\[
L\{ _0D_t^\alpha f(t) \} + aL\{ f(t) \} = L\{0\}
\]
\[
s^\alpha F(s) - \sum_{k=0}^{n-1} s^k \left[ _0D_t^{\alpha-k-1}f(t) \right]_{t=0} + aF(s) = 0
\]
For \(0 < \alpha \leq 1\), \(k\) is zero, therefore, we get
\[
s^\alpha F(s) - \left[ _0D_t^{\alpha-1}f(t) \right]_{t=0} + aF(s) = 0
\]
\[
s^\alpha F(s) - C + aF(s) = 0
\]
\[
F(s) = \frac{C}{s^\alpha + a}
\]
Taking inverse Laplace transform, we get
\[
L^{-1}\left[F(s)\right] = CL^{-1}\left[\frac{1}{s^\alpha + a}\right]
\]
\[
f(t) = Ct^{\alpha-1}E_{\alpha,\alpha}\left(-a\sqrt{t}\right)
\]  
(5.8)

where \(E_{\alpha,\alpha}\) is a Mittag-Liffler function [6].

**Test Problem** In particular, if \(\alpha = \frac{1}{2}\), then we have the following differential equation
\[
_0D_t^{1/2} f(t) + af(t) = 0, (t > 0), \left[ _0D_t^{-1/2}f(t) \right]_{t=0} = C
\]
Applying the Laplace transform on both sides, we get
\[ L[0D_t^{1/2} f(t)] + aL[f(t)] = 0 \]
\[ s^{1/2}F(s) - [0D_t^{-1/2} f(0)] + aF(s) = 0 \]
\[ s^{1/2}F(s) - C + aF(s) = 0 \]
\[ F(s) = \frac{C}{s^{1/2} + a} \]

Taking inverse Laplace transform, we get
\[ L^{-1}\left[ F(s) \right] = CL^{-1}\left[ \frac{1}{s^{1/2} + a} \right] \]
\[ f(t) = C t^{-1/2} E_{1, 1/2} \left( -a \sqrt{t} \right) \]

Note that the formula (5.8), gives us direct solution of above particular problem.

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**References**


